

AN EQUILIBRIUM PROBLEM FOR THE LIMITING EIGENVALUE DISTRIBUTION OF BANDED TOEPLITZ MATRICES

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ABSTRACT. We study the limiting eigenvalue distribution of $n \times n$ banded Toeplitz matrices as $n \rightarrow \infty$. From classical results of Schmidt-Spitzer and Hirschman it is known that the eigenvalues accumulate on a special curve in the complex plane and the normalized eigenvalue counting measure converges weakly to a measure on this curve as $n \rightarrow \infty$. In this paper, we characterize the limiting measure in terms of an equilibrium problem. The limiting measure is one component of the unique vector of measures that minimizes an energy functional defined on admissible vectors of measures. In addition, we show that each of the other components is the limiting measure of the normalized counting measure on certain generalized eigenvalues.

1. INTRODUCTION

For an integrable function $a : \{z \in \mathbb{C} \mid |z| = 1\} \rightarrow \mathbb{C}$ defined on the unit circle in the complex plane, the $n \times n$ Toeplitz matrix $T_n(a)$ with symbol a is defined by

$$(T_n(a))_{jk} = a_{j-k}, \quad j, k = 1, \dots, n, \quad (1.1)$$

where a_k is the k th Fourier coefficient of a ,

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta} d\theta. \quad (1.2)$$

In this paper we study banded Toeplitz matrices for which the symbol has only a finite number of non-zero Fourier coefficients. We assume that there exist $p, q \geq 1$ such that

$$a(z) = \sum_{k=-q}^p a_k z^k, \quad a_p \neq 0, \quad a_{-q} \neq 0. \quad (1.3)$$

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Thus $T_n(a)$ has at most $p + q + 1$ non-zero diagonals. As in [1, p. 263], we also assume without loss of generality that

$$\text{g.c.d. } \{k \in \mathbb{Z} \mid a_k \neq 0\} = 1. \quad (1.4)$$

We are interested in the limiting behavior of the spectrum of $T_n(a)$ as $n \rightarrow \infty$. We use $\text{sp } T_n(a)$ to denote the spectrum of $T_n(a)$:

$$\text{sp } T_n(a) = \{\lambda \in \mathbb{C} \mid \det(T_n(a) - \lambda I) = 0\}$$

Spectral properties of banded Toeplitz matrices are the topic of the recent book [1] by Böttcher and Grudsky. We will refer to this book frequently, in particular to Chapter 11 where the limiting behavior of the spectrum is discussed.

The limiting behavior of $\text{sp } T_n(a)$ was characterized by Schmidt and Spitzer [10]. They considered the set

$$\liminf_{n \rightarrow \infty} \text{sp } T_n(a), \quad (1.5)$$

consisting of all $\lambda \in \mathbb{C}$ such that there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, with $\lambda_n \in \text{sp } T_n(a)$, converging to λ , and the set

$$\limsup_{n \rightarrow \infty} \text{sp } T_n(a), \quad (1.6)$$

consisting of all λ such that there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, with $\lambda_n \in \text{sp } T_n(a)$, that has a subsequence converging to λ . Schmidt and Spitzer showed that these two sets are equal and can be characterized in terms of the algebraic equation

$$a(z) - \lambda = \sum_{k=-q}^p a_k z^k - \lambda = 0. \quad (1.7)$$

For every $\lambda \in \mathbb{C}$ there are $p+q$ solutions for (1.7), which we denote by $z_j(\lambda)$, for $j = 1, \dots, p+q$. We order these solutions by absolute value, so that

$$0 < |z_1(\lambda)| \leq |z_2(\lambda)| \leq \dots \leq |z_{p+q}(\lambda)|. \quad (1.8)$$

When all inequalities in (1.8) are strict then the values $z_k(\lambda)$ are unambiguously defined. If equalities occur then we choose an arbitrary numbering so that (1.8) holds. The result by Schmidt and Spitzer [10], [1, Theorem 11.17], is that

$$\liminf_{n \rightarrow \infty} \text{sp } T_n(a) = \limsup_{n \rightarrow \infty} \text{sp } T_n(a) = \Gamma_0 \quad (1.9)$$

where

$$\Gamma_0 := \{\lambda \in \mathbb{C} \mid |z_q(\lambda)| = |z_{q+1}(\lambda)|\}. \quad (1.10)$$

This result gives a description of the asymptotic location of the eigenvalues. The eigenvalues accumulate on the set Γ_0 , which is known to be a disjoint union of a finite number of (open) analytic arcs and a finite number of exceptional points [1, Theorem 11.9]. It is also known that Γ_0 is connected

[13], [1, Theorem 11.19], and that $\mathbb{C} \setminus \Gamma_0$ need not be connected [1, Theorem 11.20], [2, Proposition 5.2]. See [1] for many beautiful illustrations of eigenvalues of banded Toeplitz matrices.

The limiting eigenvalue distribution was determined by Hirschman [5], [1, Theorem 11.16]. He showed that there exists a Borel probability measure μ_0 on Γ_0 such that the normalized eigenvalue counting measure of $T_n(a)$ converges weakly to μ_0 , as $n \rightarrow \infty$. That is,

$$\frac{1}{n} \sum_{\lambda \in \text{sp } T_n(a)} \delta_\lambda \rightarrow \mu_0, \quad (1.11)$$

where in the sum each eigenvalue is counted according to its multiplicity. The measure μ_0 is absolutely continuous with respect to the arclength measure on Γ_0 and has an analytic density on each open analytic arc in Γ_0 , which can be explicitly represented in terms of the solutions of the algebraic equation (1.7) as follows. Equip every open analytic arc in Γ_0 with an orientation. The orientation induces \pm -sides on each arc, where the $+$ -side is on the left when traversing the arc according to its orientation, and the $-$ -side is on the right. The limiting measure μ_0 is then given by

$$d\mu_0(\lambda) = \frac{1}{2\pi i} \sum_{j=1}^q \left(\frac{z'_{j+}(\lambda)}{z_{j+}(\lambda)} - \frac{z'_{j-}(\lambda)}{z_{j-}(\lambda)} \right) d\lambda. \quad (1.12)$$

where $d\lambda$ is the complex line element on Γ_0 (taken according to the orientation), and where $z_{j\pm}(\lambda)$, $\lambda \in \Gamma_0$, is the limiting value of $z_j(\lambda')$ as $\lambda' \rightarrow \lambda$ from the \pm side of the arc. These limiting values exist for every $\lambda \in \Gamma_0$, with the possible exception of the finite number of exceptional points.

Note that the right-hand side of (1.12) is a priori a complex measure and it is not immediately clear that it is in fact a probability measure. In the original paper [5] and in the book [1, Theorem 11.16], the authors give a different expression for the limiting density, from which it is clear that the measure is non-negative. We prefer to work with the complex expression (1.12), since it allows for a direct generalization which we will need in this paper.

Note also that if we reverse the orientation on an arc in Γ_0 , then the \pm -sides are reversed. Since the complex line element $d\lambda$ changes sign as well, the expression (1.12) does not depend on the choice of orientation.

The following is a very simple example, which however serves as a motivation for the results in the paper.

Example 1.1. Consider the symbol $a(z) = z + 1/z$. In this case we find that $\Gamma_0 = [-2, 2]$ and μ_0 is absolutely continuous with respect to the Lebesgue measure and has density

$$\frac{d\mu_0(\lambda)}{d\lambda} = \frac{1}{\pi\sqrt{4-\lambda^2}}, \quad \lambda \in (-2, 2). \quad (1.13)$$

This measure is well-known in potential theory and is called the arcsine measure or the equilibrium measure of Γ_0 , see e.g. [9]. It has the property that it minimizes the energy functional I defined by

$$I(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y), \quad (1.14)$$

among all Borel probability measures μ on $[-2, 2]$. The measure μ_0 is also characterized by the equilibrium condition

$$\int \log |x - \lambda| d\mu_0(\lambda) = 0, \quad x \in [-2, 2], \quad (1.15)$$

which is the Euler-Lagrange variational condition for the minimization problem.

The fact that μ_0 is the equilibrium measure of Γ_0 is special for symbols a with $p = q = 1$. In that case one may think of the eigenvalues of $T_n(a)$ as charged particles on Γ_0 , each eigenvalue having a total charge $1/n$, that repel each other with logarithmic interaction. The particles seek to minimize the energy functional (1.14). As $n \rightarrow \infty$, they distribute themselves according to μ_0 and μ_0 is the minimizer of (1.14) among all probability measures supported on Γ_0 .

The aim of this paper is to characterize μ_0 for general symbols a of the form (1.3) also in terms of an equilibrium problem from potential theory. The corresponding equilibrium problem is more complicated since it involves not only the measure μ_0 , but a sequence of $p + q - 1$ measures

$$\mu_{-q+1}, \mu_{-q+2}, \dots, \mu_{-1}, \mu_0, \mu_1, \dots, \mu_{p-2}, \mu_{p-1}$$

that jointly minimize an energy functional.

2. STATEMENT OF RESULTS

2.1. The energy functional. To state our results we need to introduce some notions from potential theory. Main references for potential theory in the complex plane are [8] and [9].

We will mainly work with finite positive measures on \mathbb{C} , but we will also use $\nu_1 - \nu_2$ where ν_1 and ν_2 are positive measures. The measures need not have bounded support. If ν has unbounded support then we assume that

$$\int \log(1 + |x|) d\nu(x) < \infty. \quad (2.1)$$

In that case the logarithmic energy of ν is defined as

$$I(\nu) = \iint \log \frac{1}{|x-y|} d\nu(x) d\nu(y) \quad (2.2)$$

and $I(\nu) \in (-\infty, +\infty]$.

Definition 2.1. We define \mathcal{M}_e as the collection of positive measures ν on \mathbb{C} satisfying (2.1) and having finite energy, i.e., $I(\nu) < +\infty$. For $c > 0$ we define

$$\mathcal{M}_e(c) = \{\nu \in \mathcal{M}_e \mid \nu(\mathbb{C}) = c\}. \quad (2.3)$$

The mutual energy $I(\nu_1, \nu_2)$ of two measures ν_1 and ν_2 is

$$I(\nu_1, \nu_2) = \int \log \frac{1}{|x - y|} d\nu_1(x) d\nu_2(y). \quad (2.4)$$

It is well-defined and finite if $\nu_1, \nu_2 \in \mathcal{M}_e$ and in that case we have

$$I(\nu_1 - \nu_2) = I(\nu_1) + I(\nu_2) - 2I(\nu_1, \nu_2). \quad (2.5)$$

If $\nu_1, \nu_2 \in \mathcal{M}_e(c)$ for some $c > 0$, then

$$I(\nu_1 - \nu_2) \geq 0, \quad (2.6)$$

with equality if and only if $\nu_1 = \nu_2$. This is a well-known result if ν_1 and ν_2 have compact support [9]. For measures in $\mathcal{M}_e(c)$ with unbounded support, this is a recent result of Simeonov [11], who obtained this from a very elegant integral representation for $I(\nu_1 - \nu_2)$. It is a consequence of (2.6) that I is strictly convex on $\mathcal{M}_e(c)$, since

$$\begin{aligned} I\left(\frac{\nu_1 + \nu_2}{2}\right) &= \frac{1}{2}(I(\nu_1) + I(\nu_2)) - I\left(\frac{\nu_1 - \nu_2}{2}\right) \\ &\leq \frac{1}{2}(I(\nu_1) + I(\nu_2)), \end{aligned} \quad \text{for } \nu_1, \nu_2 \in \mathcal{M}_e(c),$$

with equality if and only if $\nu_1 = \nu_2$.

Before we can state the equilibrium problem we also need to introduce the sets

$$\Gamma_k := \{\lambda \in \mathbb{C} \mid |z_{q+k}(\lambda)| = |z_{q+k+1}(\lambda)|\}, \quad k = -q + 1, \dots, p - 1, \quad (2.7)$$

which for $k = 0$ reduces to the definition (1.10) of Γ_0 . We will show that each Γ_k is the disjoint union of a finite number of open analytic arcs and a finite number of exceptional points. All Γ_k are unbounded, except for Γ_0 which is compact.

The equilibrium problem will be defined for a vector of measures denoted by $\vec{\nu} = (\nu_{-q+1}, \dots, \nu_{p-1})$. The component ν_k is a measure on Γ_k satisfying some additional properties that are given in the following definition.

Definition 2.2. We call a vector of measures $\vec{\nu} = (\nu_{-q+1}, \dots, \nu_{p-1})$ admissible if $\nu_k \in \mathcal{M}_e$, ν_k is supported on Γ_k , and

$$\nu_k(\Gamma_k) = \begin{cases} \frac{q+k}{q} & \text{if } k \leq 0, \\ \frac{p-k}{p} & \text{if } k \geq 0, \end{cases} \quad (2.8)$$

for every $k = -q + 1, \dots, p - 1$.

Now we are ready to state our first result. The proof is given in section 4.

Theorem 2.3. *Let the symbol a satisfy (1.3) and (1.4), and let the curves Γ_k be defined as in (2.7). For each $k \in \{-q+1, \dots, p-1\}$, define the measure μ_k on Γ_k by*

$$d\mu_k(\lambda) = \frac{1}{2\pi i} \sum_{j=1}^{q+k} \left(\frac{z'_{j+}(\lambda)}{z_{j+}(\lambda)} - \frac{z'_{j-}(\lambda)}{z_{j-}(\lambda)} \right) d\lambda, \quad (2.9)$$

where $d\lambda$ is the complex line element on each analytic arc of Γ_k according to a chosen orientation of Γ_k (cf. discussion after (1.12)). Then

- (a) $\vec{\mu} = (\mu_{-q+1}, \dots, \mu_{p-1})$ is admissible.
- (b) There exist constants l_k such that

$$2 \int \log |\lambda - x| d\mu_k(x) = \int \log |\lambda - x| d\mu_{k+1}(x) + \int \log |\lambda - x| d\mu_{k-1}(x) + l_k, \quad (2.10)$$

for $k = -q+1, \dots, p-1$, and $\lambda \in \Gamma_k$. Here we let μ_{-q} and μ_p be the zero measures.

- (c) $\vec{\mu} = (\mu_{-q+1}, \dots, \mu_{p-1})$ is the unique minimizer of the energy functional J defined by

$$J(\vec{\nu}) = \sum_{k=-q+1}^{p-1} I(\nu_k) - \sum_{k=-q+1}^{p-2} I(\nu_k, \nu_{k+1}) \quad (2.11)$$

for admissible vectors of measures $\vec{\nu} = (\nu_{-q+1}, \dots, \nu_{p-1})$.

The relations (2.10) are the Euler-Lagrange variational conditions for the minimization problem for J among admissible vectors of measures.

It may not be obvious that the energy functional (2.11) is bounded from below. This can be seen from the alternative representation

$$\begin{aligned} J(\vec{\nu}) &= \left(\frac{1}{q} + \frac{1}{p} \right) I(\nu_0) + \sum_{k=1}^{q-1} k(k+1) I\left(\frac{\nu_{-q+k}}{k} - \frac{\nu_{-q+k+1}}{k+1} \right) \\ &\quad + \sum_{k=1}^{p-1} k(k+1) I\left(\frac{\nu_{p-k}}{k} - \frac{\nu_{p-k-1}}{k+1} \right). \end{aligned} \quad (2.12)$$

We leave the calculation leading to this identity to the reader. Under the normalizations (2.8) it follows by (2.6) that each term in the two finite sums on the right-hand side of (2.12) is non-negative, so that

$$J(\vec{\nu}) \geq \left(\frac{1}{q} + \frac{1}{p} \right) I(\nu_0).$$

Since ν_0 is a Borel probability measure on Γ_0 and Γ_0 is compact, we indeed have that the energy functional is bounded from below on admissible vectors of measures $\vec{\nu}$.

The alternative representation (2.12) will play a role in the proof of Theorem 2.3.

Yet another representation for J is

$$J(\vec{\nu}) = \sum_{j,k=-q+1}^{p-1} A_{jk} I(\nu_j, \nu_k) \quad (2.13)$$

where the interaction matrix A has entries

$$A_{jk} = \begin{cases} 1, & \text{if } j = k, \\ -\frac{1}{2}, & \text{if } |j - k| = 1, \\ 0, & \text{if } |j - k| \geq 2. \end{cases} \quad (2.14)$$

The energy functional in the form (2.13) and (2.14) also appears in the theory of simultaneous rational approximation, where it is the interaction matrix for a Nikishin system [7, Chapter 5].

It allows for the following physical interpretation: on each of the curves Γ_k one puts charged particles with total charge $(q+k)/q$ or $(p-k)/p$, depending on whether $k \leq 0$ or $k \geq 0$. Particles that lie on the same curve repel each other. The particles on two consecutive curves interact in the sense that they attract each other but in a way that is half as strong as the repulsion on a single curve. Particles on different curves that are not consecutive do not interact with each other in a direct way.

2.2. The measures μ_k as limiting measures of generalized eigenvalues. By (1.12) and Theorem 2.3 we know that the measure μ_0 that appears in the minimizer of the energy functional J is the limiting measure for the eigenvalues of $T_n(a)$. It is natural to ask about the other measures μ_k that appear in the minimizer. In our second result we show that the measures μ_k can be obtained as limiting counting measures for certain generalized eigenvalues.

Let $k \in \{-q+1, \dots, p-1\}$. We use $T_n(z^{-k}(a-\lambda))$ to denote the Toeplitz matrix with the symbol $z \mapsto z^{-k}(a(z) - \lambda)$. For example, for $k = 1$, $q = 1$ and $p = 2$, we have

$$T_n(z^{-k}(a-\lambda)) = \begin{pmatrix} a_1 & a_0 - \lambda & a_{-1} & & & & \\ a_2 & a_1 & a_0 - \lambda & a_{-1} & & & \\ & a_2 & a_1 & a_0 - \lambda & a_{-1} & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & a_2 & a_1 & a_0 - \lambda & a_{-1} \\ & & & & a_2 & a_1 & a_0 - \lambda \\ & & & & & a_2 & a_1 \end{pmatrix}_{n \times n}.$$

Definition 2.4. For $k \in \{-q+1, \dots, p-1\}$ and $n \geq 1$, we define the polynomial $P_{k,n}$ by

$$P_{k,n}(\lambda) = \det T_n(z^{-k}(a-\lambda)) \quad (2.15)$$

and we define the k th generalized spectrum of $T_n(a)$ by

$$\text{sp}_k T_n(a) = \{\lambda \in \mathbb{C} \mid P_{k,n}(\lambda) = 0\}. \quad (2.16)$$

Finally, we define $\mu_{k,n}$ as the normalized zero counting measure of $\text{sp}_k T_n(a)$

$$\mu_{k,n} = \frac{1}{n} \sum_{\lambda \in \text{sp}_k T_n(a)} \delta_\lambda \quad (2.17)$$

where in the sum each λ is counted according to its multiplicity as a zero of $P_{k,n}$.

Note that $\lambda \in \text{sp}_k T_n(a)$ is a generalized eigenvalue (in the usual sense) for the matrix pencil $(T_n(z^{-k}a), T_n(z^{-k}))$, that is, $\det(A - \lambda B) = 0$ with $A = T_n(z^{-k}a)$ and $B = T_n(z^{-k})$. If $k = 0$, then $B = I$ and $\text{sp}_0 T_n(a) = \text{sp } T_n(a)$. If $k \neq 0$, then B is not invertible and the generalized eigenvalue problem is singular, causing that there are less than n generalized eigenvalues. In fact, since $T_n(z^{-k}(a - \lambda))$ has exactly $n - |k|$ entries $a_0 - \lambda$, we easily get that the degree of $P_{k,n}$ is at most $n - |k|$ and so there are at most $n - |k|$ generalized eigenvalues. Due to the band structure of $T_n(z^{-k}(a - \lambda))$ the actual number of generalized eigenvalues is substantially smaller.

Proposition 2.5. *Let $k \in \{-q + 1, \dots, p - 1\}$. Let $P_{k,n}(\lambda) = \gamma_{k,n} \lambda^{d_{k,n}} + \dots$ have degree $d_{k,n}$ and leading coefficient $\gamma_{k,n} \neq 0$. Then*

$$d_{k,n} \leq \begin{cases} \frac{q+k}{q}n, & \text{if } k < 0, \\ \frac{p-k}{p}n, & \text{if } k > 0. \end{cases} \quad (2.18)$$

Equality holds in (2.18) if either $k > 0$ and n is a multiple of p , or $k < 0$ and n is a multiple of q , and in those cases we have

$$\gamma_{k,n} = \begin{cases} (-1)^{(k+1)n} a_{-q}^{|k|n/q}, & \text{if } k < 0 \text{ and } n \equiv 0 \pmod{q}, \\ (-1)^{(k+1)n} a_p^{kn/p}, & \text{if } k > 0 \text{ and } n \equiv 0 \pmod{p}. \end{cases} \quad (2.19)$$

We now come to our second main result. It is the analogue of the results of Schmidt-Spitzer and Hirschman for the generalized eigenvalues.

Theorem 2.6. *Let $k \in \{-q + 1, \dots, p - 1\}$. Then*

$$\liminf_{n \rightarrow \infty} \text{sp}_k T_n(a) = \limsup_{n \rightarrow \infty} \text{sp}_k T_n(a) = \Gamma_k, \quad (2.20)$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{C}} \phi(z) d\mu_{k,n}(z) = \int_{\mathbb{C}} \phi(z) d\mu_k(z) \quad (2.21)$$

holds for every bounded continuous function ϕ on \mathbb{C} .

The key element in the proof of Theorem 2.6 is a beautiful formula of Widom [14], see [1, Theorem 2.8], for the determinant of a banded Toeplitz matrix. In the present situation Widom's formula yields the following. Let $\lambda \in \mathbb{C}$ be such that the solutions $z_j(\lambda)$ of the algebraic equation (1.7) are mutually distinct. Then

$$P_{k,n}(\lambda) = \det T_n(z^{-k}(a - \lambda)) = \sum_M C_M(\lambda) (w_M(\lambda))^n, \quad (2.22)$$

where the sum is over all subsets $M \subset \{1, 2, \dots, p+q\}$ of cardinality $|M| = p-k$ and for each such M , we have

$$w_M(\lambda) := (-1)^{p-k} a_p \prod_{j \in M} z_j(\lambda), \quad (2.23)$$

and (with $\overline{M} := \{1, 2, \dots, p+q\} \setminus M$),

$$C_M(\lambda) := \prod_{j \in M} z_j(\lambda)^{q+k} \prod_{\substack{j \in M \\ l \in \overline{M}}} (z_j(\lambda) - z_l(\lambda))^{-1}. \quad (2.24)$$

The formula (2.22) shows that for large n , the main contribution comes from those M for which $|w_M(\lambda)|$ is the largest possible. For $\lambda \in \mathbb{C} \setminus \Gamma_k$ there is a unique such M , namely

$$M = M_k := \{q+k+1, q+k+2, \dots, p+q\} \quad (2.25)$$

because of the ordering (1.8).

2.3. Overview of the rest of the paper. In section 3 we will state some preliminary results about analyticity properties of the solutions z_j of the algebraic equation (1.7). These results will be needed in the proof of Theorem 2.3 which is given in section 4. In section 5 we will prove Proposition 2.5 and Theorem 2.6. Finally, we conclude the paper by giving some examples in section 6.

3. PRELIMINARIES

In this section we collect a number of properties of the curves Γ_k and the solutions $z_1(\lambda), \dots, z_{p+q}(\lambda)$ of the algebraic equation (1.7). For convenience we define throughout the rest of the paper

$$\Gamma_{-q} = \Gamma_p = \emptyset, \quad \text{and} \quad \mu_{-q} = \mu_p = 0. \quad (\text{the zero-measure}).$$

Occasionally we also use

$$z_0(\lambda) = 0, \quad z_{p+q+1}(\lambda) = +\infty.$$

3.1. The structure of the curves Γ_k . We start with a definition, cf. [1, §11.2].

Definition 3.1. *A point $\lambda_0 \in \mathbb{C}$ is called a branch point if $a(z) - \lambda_0 = 0$ has a multiple root. A point $\lambda_0 \in \Gamma_k$ is an exceptional point of Γ_k if λ_0 is a branch point, or if there is no open neighborhood U of λ such that $\Gamma_k \cap U$ is an analytic arc starting and terminating on ∂U .*

If λ_0 is a branch point, then there is a z_0 such that $a(z_0) = \lambda_0$ and $a'(z_0) = 0$. Then we may assume that $z_0 = z_{q+k}(\lambda_0) = z_{q+k+1}(\lambda_0)$ for some k and $\lambda_0 \in \Gamma_k$. For a symbol a of the form (1.3), the derivative a' has exactly $p+q$ zeros (counted with multiplicity), so that there are exactly $p+q$ branch points counted with multiplicity.

The solutions $z_k(\lambda)$ also have branching at infinity (unless $p = 1$ or $q = 1$). There are p solutions of (1.7) that tend to infinity as $\lambda \rightarrow \infty$, and q solutions that tend to 0. Indeed, we have

$$z_k(\lambda) = \begin{cases} c_k \lambda^{-1/q} (1 + \mathcal{O}(\lambda^{-1/q})), & \text{for } k = 1, \dots, q, \\ c_k \lambda^{1/p} (1 + \mathcal{O}(\lambda^{-1/p})), & \text{for } k = q+1, \dots, p+q, \end{cases} \quad (3.1)$$

as $\lambda \rightarrow \infty$. Here c_1, \dots, c_q are the q distinct solutions of $c^q = a_{-q}$ (taken in some order depending on λ), and c_{q+1}, \dots, c_{p+q} are the p distinct solutions of $c^p = a_p^{-1}$ (again taken in some order depending on λ).

The following proposition gives the structure of Γ_k at infinity.

Proposition 3.2. *Let $k \in \{-q+1, \dots, p-1\} \setminus \{0\}$. Then there is an $R > 0$ such that $\Gamma_k \cap \{\lambda \in \mathbb{C} \mid |\lambda| > R\}$ is a finite disjoint union of analytic arcs, each extending from $|\lambda| = R$ to infinity.*

Proof. The proof is similar to the proof of [1, Proposition 11.8] where a similar structure theorem was proved for finite branch points. We omit the details. \square

It follows from Proposition 3.2 that the exceptional points for Γ_k are in a bounded set. Since the set of exceptional point is discrete we conclude that there are only finitely many exceptional points. Then we have the following result about the structure of Γ_k .

Proposition 3.3. *For every $k \in \{-q+1, \dots, p-1\}$, the set Γ_k is the disjoint union of a finite number of open analytic arcs and a finite number of exceptional points. The set Γ_k has no isolated points.*

Proof. This was proved for $k = 0$ in [10] and [1, Theorem 11.9]. For general k , there are only finitely many exceptional points and the proof follows in a similar way. \square

3.2. The Riemann surface. From Proposition 3.3 it follows that the curves Γ_k can be taken as cuts for the $p+q$ -sheeted Riemann surface of the algebraic equation (1.7). We number the sheets from 1 to $p+q$, where the k th sheet of the Riemann surface is

$$\mathcal{R}_k = \{\lambda \in \mathbb{C} \mid |z_{k-1}(\lambda)| < |z_k(\lambda)| < |z_{k+1}(\lambda)|\} = \mathbb{C} \setminus (\Gamma_{-q+k-1} \cup \Gamma_{-q+k}). \quad (3.2)$$

Thus z_k is well-defined and analytic on \mathcal{R}_k .

The easiest case to visualize is the case where consecutive cuts are disjoint, that is, $\Gamma_{-q+k-1} \cap \Gamma_{-q+k} = \emptyset$ for every $k = 2, \dots, p+q-2$. In that case we have that \mathcal{R}_k is connected to \mathcal{R}_{k+1} via Γ_{-q+k} in the usual crosswise manner, and z_{k+1} is the analytic continuation of z_k across Γ_{-q+k} .

The general case is described in the following proposition.

Proposition 3.4. *Suppose A is an open analytic arc such that $A \subset \Gamma_{-q+k}$, for $k = k_1, \dots, k_2$, and $A \cap (\Gamma_{-q+k_1-1} \cup \Gamma_{-q+k_2+1}) = \emptyset$. Then for $k =$*

$k_1, \dots, k_2 + 1$, we have that the analytic continuation of z_k across A is equal to $z_{k_1+k_2-k+1}$. Thus across A , we have that \mathcal{R}_k is connected to $\mathcal{R}_{k_1+k_2-k+1}$.

Proof. We have that

$$|z_{k_1}(\lambda)| = |z_{k_1+1}(\lambda)| = \dots = |z_{k_2}(\lambda)| = |z_{k_2+1}(\lambda)|$$

for $\lambda \in A$, with strict inequalities ($<$) for λ on either side of A . Choose an orientation for A . Then there is a permutation π of $\{k_1, \dots, k_2 + 1\}$ such that $z_{\pi(k)}$ is the analytic continuation of z_k from the $+$ -side of A to the $-$ -side of A .

Assume that there are $k, k' \in \{k_1, \dots, k_2 + 1\}$ such that $k < k'$ and $\pi(k) < \pi(k')$. Take a regular $\lambda_0 \in A$ and a small neighborhood U of λ_0 such that $A \cap U = \Gamma_{-q+k} \cap U = \Gamma_{-q+k'} \cap U$ and $A \cap U$ is an analytic arc starting and terminating on ∂U . Then we have a disjoint union $U = U_+ \cup U_- \cup (A \cap U)$ where U_+ (U_-) is the part of U on the $+$ -side ($-$ -side) of A . The function ϕ defined by

$$\phi(\lambda) = \begin{cases} \frac{z_k(\lambda)}{z_{k'}(\lambda)}, & \text{for } \lambda \in U_+, \\ \frac{z_{\pi(k)}(\lambda)}{z_{\pi(k')}(\lambda)}, & \text{for } \lambda \in U_-, \end{cases}$$

has an analytic continuation to U , and satisfies $|\phi(\lambda)| < 1$ for $\lambda \in U_+ \cup U_-$ and $|\phi(\lambda)| = 1$ for $\lambda \in A \cap U$. This contradicts the maximum principle for analytic functions. Therefore $\pi(k) > \pi(k')$ for every $k, k' \in \{k_1, \dots, k_2 + 1\}$ with $k < k'$, and this implies that $\pi(k) = k_1 + k_2 - k + 1$ for every $k = k_1, \dots, k_2 + 1$, and the proposition follows. \square

3.3. The functions $w_k(\lambda)$. A major role is played by the functions w_k , which for $k \in \{-q + 1, \dots, p - 1\}$, are defined by

$$w_k(\lambda) = \prod_{j=1}^{q+k} z_j(\lambda), \quad \text{for } \lambda \in \mathbb{C} \setminus \Gamma_k. \quad (3.3)$$

Note that $w_k = (-1)^{p-k} a_p^{-1} w_{\{1, \dots, k\}}$ in the notation of (2.23).

Proposition 3.5. *The function w_k is analytic in $\mathbb{C} \setminus \Gamma_k$.*

Proof. Since z_j is analytic on $\mathcal{R}_j = \mathbb{C} \setminus (\Gamma_{-q+j-1} \cup \Gamma_{-q+j})$, see (3.2), we obtain from its definition that w_k is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{k+q} \Gamma_{-q+j}$. Let A be an analytic arc in $\Gamma_{-q+j} \setminus \Gamma_k$ for some $j < k + q$. Choose an orientation on A . Since the arc is disjoint from Γ_k , we have that $z_{j+}(\lambda) = z_{\pi(j)-}(\lambda)$, for $\lambda \in A$ and $j = 1, \dots, q + k$, where π is a permutation of $\{1, \dots, q + k\}$. Since w_k is symmetric in the z_j 's for $j = 1, \dots, q + k$, it then follows that

$$w_{k+}(\lambda) = w_{k-}(\lambda), \quad \text{for } \lambda \in A,$$

which shows the analyticity in $\mathbb{C} \setminus \Gamma_k$ with the possible exception of isolated singularities at the exceptional points of $\Gamma_{-q+1}, \Gamma_{-q+2}, \dots, \Gamma_{k-1}$. However, each z_j , and therefore also w_k , is bounded near such an exceptional point, so that any isolated singularity is removable. \square

In the rest of the paper we make frequently use of the logarithmic derivative w'_k/w_k of w_k . By the fact that w_k does not vanish on $\mathbb{C} \setminus \Gamma_k$ and Proposition 3.5, it follows that w'_k/w_k is analytic in $\mathbb{C} \setminus \Gamma_k$. By Proposition 3.4 it moreover has an analytic continuation across every open analytic arc $A \subset \Gamma_k$. Near the exceptional points that are no branch points w'_k/w_k remains bounded. At the branch points it can however have singularities of a certain order.

Proposition 3.6. *Let $\lambda_0 \in \Gamma_k$ be a branch point of Γ_k . Then there exists an $m \in \mathbb{N}$ such that*

$$\frac{w'_k(\lambda)}{w_k(\lambda)} = \mathcal{O}\left((\lambda - \lambda_0)^{-m/(m+1)}\right), \quad (3.4)$$

as $\lambda \rightarrow \lambda_0$ with $\lambda \in \mathbb{C} \setminus \Gamma_k$.

Proof. Let $1 \leq j \leq q+k$. We investigate the behavior of $z_j(\lambda)$ when $\lambda \rightarrow \lambda_0$ such that λ remains in a connected component of $\mathbb{C} \setminus (\Gamma_{j-1} \cup \Gamma_j)$. Then $z_j(\lambda) \rightarrow z_0$ for some $z_0 \in \mathbb{C}$ with $a(z_0) = \lambda_0$. Let $m_0 + 1$ be the multiplicity of z_0 as a solution of $a(z) = \lambda_0$. Then

$$a(z) = \lambda_0 + c_0(z - z_0)^{m_0+1}(1 + \mathcal{O}(z - z_0)), \quad z \rightarrow z_0, \quad (3.5)$$

for some nonzero constant c_0 . Therefore,

$$z_j(\lambda) = z_0 + \mathcal{O}((\lambda - \lambda_0)^{1/(m_0+1)}), \quad (3.6)$$

and

$$z'_j(\lambda) = \mathcal{O}((\lambda - \lambda_0)^{-m_0/(m_0+1)}), \quad (3.7)$$

for $\lambda \rightarrow \lambda_0$ such that λ remains in the same connected component of $\mathbb{C} \setminus (\Gamma_{j-1} \cup \Gamma_j)$. Let m be the maximum of all the multiplicities of the roots of $a(z) = \lambda_0$. Then it follows from (3.6) and (3.7) that

$$\frac{z'_j(\lambda)}{z_j(\lambda)} = \mathcal{O}((\lambda - \lambda_0)^{-m/(m+1)})$$

as $\lambda \rightarrow \lambda_0$ with $\lambda \in \mathbb{C} \setminus \Gamma_k$. Then we obtain (3.4) in view of (3.3). \square

We end this section by giving the asymptotics of w'_k/w_k for $\lambda \rightarrow \infty$.

Proposition 3.7. *As $\lambda \rightarrow \infty$ with $\lambda \in \mathbb{C} \setminus \Gamma_k$, we have*

$$\frac{w'_k(\lambda)}{w_k(\lambda)} = \begin{cases} -\frac{q+k}{q}\lambda^{-1} + \mathcal{O}(\lambda^{-1-1/q}), & \text{for } k = -q+1, \dots, -1, \\ -\lambda^{-1} + \mathcal{O}(\lambda^{-2}), & \text{for } k = 0, \\ -\frac{p-k}{p}\lambda^{-1} + \mathcal{O}(\lambda^{-1-1/p}), & \text{for } k = 1, \dots, p-1. \end{cases} \quad (3.8)$$

Proof. This follows directly from (3.1) and (3.3). \square

4. PROOF OF THEOREM 2.3

We use the function w_k introduced in (3.3). We define μ_k by the formula (2.9) and we note that

$$d\mu_k(\lambda) = \frac{1}{2\pi i} \left(\frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} \right) d\lambda. \quad (4.1)$$

Proposition 4.1. *For each $k = -q + 1, \dots, p - 1$, we have that μ_k is a measure on Γ_k with total mass $\mu_k(\Gamma_k) = (q + k)/q$ if $k \geq 0$, and $\mu_k(\Gamma_k) = (p - k)/p$ if $k \leq 0$.*

Proof. We first show that μ_k is a measure, i.e., that it is non-negative on each analytic arc of Γ_k . Let A be an analytic arc in Γ_k consisting only of regular points. Let $t \mapsto \lambda(t)$ be a parametrization of A in the direction of the orientation of Γ_k . Then

$$\begin{aligned} d\mu_k(\lambda) &= \frac{1}{2\pi i} \left(\frac{w'_{k+}(\lambda(t))}{w_{k+}(\lambda(t))} - \frac{w'_{k-}(\lambda(t))}{w_{k-}(\lambda(t))} \right) \lambda'(t) dt \\ &= \frac{1}{2\pi i} \left(\frac{d}{dt} \log \frac{w_{k+}(\lambda(t))}{w_{k-}(\lambda(t))} \right) dt. \end{aligned}$$

To conclude that μ_k is non-negative on A , it is thus enough to show that

$$\operatorname{Re} \log \frac{w_{k+}(\lambda)}{w_{k-}(\lambda)} = 0, \quad \text{for } \lambda \in A, \quad (4.2)$$

and

$$\operatorname{Im} \log \frac{w_{k+}(\lambda)}{w_{k-}(\lambda)} \quad \text{increases along } A. \quad (4.3)$$

Since $|w_{k+}(\lambda)| = |w_{k-}(\lambda)|$ for $\lambda \in A$, we have (4.2) so that it only remains to prove (4.3).

There is a neighborhood U of A such that $U \setminus \Gamma_k$ has two components, denoted U_+ and U_- , where U_+ is on the $+$ -side of Γ_k and U_- on the $-$ -side. It follows from Proposition 3.4 that w_k has an analytic continuation from U_- to U , which we denote by \hat{w}_k , and that $|w_k(\lambda)| < |\hat{w}_k(\lambda)|$ for $\lambda \in U_+$, and equality $|w_{k+}(\lambda)| = |\hat{w}_k(\lambda)|$ holds for $\lambda \in A$. Thus it follows that

$$\frac{\partial}{\partial n} \operatorname{Re} \log \left(\frac{w_k(\lambda)}{\hat{w}_k(\lambda)} \right) \leq 0, \quad \text{for } \lambda \in A,$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative to A in the direction of U_+ . Then by the Cauchy-Riemann equations we have that $\operatorname{Im} \log \left(\frac{w_{k+}(\lambda)}{\hat{w}_{k+}(\lambda)} \right)$ is increasing along A . Since $\hat{w}_{k+}(\lambda) = w_{k-}(\lambda)$ for $\lambda \in A$, we obtain (4.3). Thus μ_k is a measure.

Next we show that μ_k is a finite measure, which means that we have to show that

$$\frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} \quad (4.4)$$

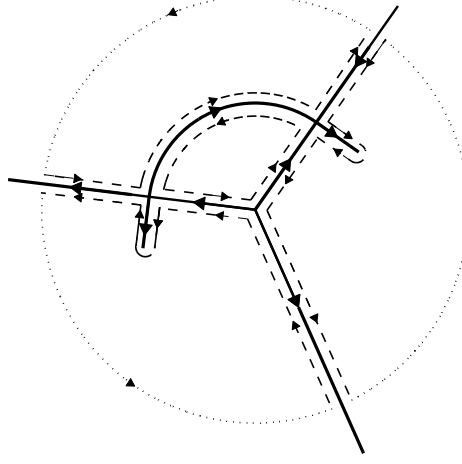


FIGURE 1. Illustration for the proofs of Propositions 4.1 and 4.2. The solid line is a sketch of a possible contour Γ_k . The dashed line is the contour $\tilde{\Gamma}_{k,R}$ and the dotted line is the boundary of a disk of radius R around 0.

is integrable near infinity on Γ_k and near every branch point on Γ_k . This follows from Propositions 3.7 and 3.6. Indeed, from Proposition 3.7 it follows that

$$\frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} = \mathcal{O}\left(\lambda^{-1-\delta}\right) \quad \text{as } \lambda \rightarrow \infty, \lambda \in \Gamma_k. \quad (4.5)$$

where $\delta = 1/q$ if $k < 0$ and $\delta = 1/p$ if $k > 0$. Since $\delta > 0$ we see that (4.4) is integrable near infinity. For a branch point λ_0 of Γ_k , we have from Proposition 3.6 that there exist an $m \geq 1$ such that

$$\frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} = \mathcal{O}\left((\lambda - \lambda_0)^{-m/(m+1)}\right) \quad \text{as } \lambda \rightarrow \lambda_0, \lambda \in \Gamma_k. \quad (4.6)$$

This shows that (4.4) is integrable near every branch point. Thus μ_k is a finite measure.

Finally we compute the total mass of μ_k . Let $D(0, R) = \{z \in \mathbb{C} \mid |z| < R\}$. Then for R large enough, so that $D(0, R)$ contains all exceptional points of Γ_k and all connected components of $\mathbb{C} \setminus \Gamma_k$ (if any),

$$\mu_k(\Gamma_k \cap D(0, R)) = \frac{1}{2\pi i} \left(\int_{\Gamma_k \cap D(0, R)} \frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} d\lambda - \int_{\Gamma_k \cap D(0, R)} \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} d\lambda \right) \quad (4.7)$$

where we have used the behavior (4.6) near the branch points in order to be able to split the integrals. Again using (4.6) we can then turn the two integrals into a contour integral over a contour $\tilde{\Gamma}_{k,R}$ as in Figure 1. The contour $\tilde{\Gamma}_{k,R}$ passes along the \pm -sides of $\Gamma_k \cap D(0, R)$ and if we choose the orientation that is also shown in Figure 1 (and which is independent of the

choice of orientation for Γ_k), then

$$\mu_k(\Gamma_k \cap D(0, R)) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{k,R}} \frac{w'_k(\lambda)}{w_k(\lambda)} d\lambda. \quad (4.8)$$

The parts of $\tilde{\Gamma}_{k,R}$ that belong to bounded components of $\mathbb{C} \setminus \Gamma_k$ form closed contours along the boundary of each bounded component. By Cauchy's theorem their contribution to the integral (4.8) vanishes. The parts of $\tilde{\Gamma}_{k,R}$ that belong to the unbounded components of $\mathbb{C} \setminus \Gamma_k$ can be deformed to the circle $\partial D(0, R)$ with the clockwise orientation. Thus if we use the positive orientation on $\partial D(0, R)$ as in Figure 1, then we obtain from (4.8)

$$\mu_k(\Gamma_k \cap D(0, R)) = -\frac{1}{2\pi i} \oint_{\partial D(0,R)} \frac{w'_k(\lambda)}{w_k(\lambda)} d\lambda$$

Letting $R \rightarrow \infty$ and using Proposition 3.7, we then find that μ_k is a measure on Γ_k with total mass $\mu_k(\Gamma_k) = (q+k)/q$ if $k \leq 0$, and $\mu_k(\Gamma_k) = (p-k)/p$ if $k \geq 0$. \square

The following proposition is the next step in showing that the measures μ_k from (2.9) satisfy the equations (2.10).

Proposition 4.2. *For $k = -q+1, \dots, p-1$, we have that*

$$\int \frac{d\mu_k(x)}{x-\lambda} = \frac{w'_k(\lambda)}{w_k(\lambda)}, \quad \text{for } \lambda \in \mathbb{C} \setminus \Gamma_k, \quad (4.9)$$

and

$$\int \log |\lambda - x| d\mu_k(x) = -\log |w_k(\lambda)| + \alpha_k, \quad \text{for } \lambda \in \mathbb{C}, \quad (4.10)$$

where α_k is the constant

$$\alpha_k = \begin{cases} \log |a_{-q}| + \frac{k}{q} \log |a_{-q}|, & \text{if } k \leq 0, \\ \log |a_{-q}| - \frac{k}{p} \log |a_p|, & \text{if } k \geq 0. \end{cases} \quad (4.11)$$

Proof. To prove (4.9), we follow the same arguments as in the calculation of $\mu_k(\Gamma_k)$ in the end of the proof of Proposition 4.1. Let $\lambda \in \mathbb{C} \setminus \Gamma_k$, and choose $R > 0$ as in the proof of Proposition 4.1. We may assume $R > |\lambda|$. Then similar to (4.7) and (4.8) we can write

$$\int_{\Gamma_k \cap D(0,R)} \frac{d\mu_k(x)}{x-\lambda} = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{k,R}} \frac{w'_k(x)}{w_k(x)(x-\lambda)} dx$$

where $\tilde{\Gamma}_{k,R}$ has the same meaning as in the proof of Proposition 4.1, see also Figure 1. As in the proof of Proposition 4.1 we deform to an integral over $\partial D(0, R)$, but now we have to take into account that the integrand has a pole at $x = \lambda$ with residue $w'_k(\lambda)/w_k(\lambda)$. Therefore, by Cauchy's theorem

$$\int_{\Gamma_k \cap D(0,R)} \frac{d\mu_k(x)}{x-\lambda} = \frac{w'_k(\lambda)}{w_k(\lambda)} - \frac{1}{2\pi i} \int_{\partial D(0,R)} \frac{w'_k(x)}{w_k(x)(x-\lambda)} dx. \quad (4.12)$$

Letting $R \rightarrow \infty$ and using Proposition 3.7 gives (4.9).

Next we integrate (4.9) over a Jordan curve J in $\mathbb{C} \setminus \Gamma_k$ from λ_1 to λ_2 .

$$\begin{aligned} \int_{\lambda_1}^{\lambda_2} \int_{\Gamma_k} \frac{1}{x - \lambda} d\mu_k(x) d\lambda &= - \int \int_{\lambda_1}^{\lambda_2} \frac{1}{x - \lambda} d\lambda d\mu_k(x) \\ &= \int (\log |\lambda_1 - x| - \log |\lambda_2 - x| + i\Delta_J[\arg(\lambda - x)]) d\mu_k(x), \end{aligned} \quad (4.13)$$

where $\Delta_J[\arg(\lambda - x)]$ denotes the change in argument of $\lambda - x$ as when λ varies over J from λ_1 to λ_2 . By (4.9) the integral (4.13) is equal to

$$\int_{\lambda_1}^{\lambda_2} \frac{w'_k(\lambda)}{w_k(\lambda)} d\lambda = \log |w_k(\lambda_2)| - \log |w_k(\lambda_1)| + i\Delta_J[\arg w_k(\lambda)]. \quad (4.14)$$

Equating the real parts of (4.13) and (4.14) we get

$$\int (\log |\lambda_1 - x| - \log |\lambda_2 - x|) d\mu_k(x) = -\log |w_k(\lambda_1)| + \log |w_k(\lambda_2)|. \quad (4.15)$$

Since λ_1 and λ_2 can be taken arbitrarily in a connected component of $\mathbb{C} \setminus \Gamma_k$, we find that there exists a constant $\alpha_k \in \mathbb{R}$ (which a priori could depend on the connected component) such that

$$\int \log |\lambda - x| d\mu_k(x) = -\log |w_k(\lambda)| + \alpha_k, \quad (4.16)$$

for all λ in a connected component of $\mathbb{C} \setminus \Gamma_k$. By continuity the equation (4.16) extends to the closure of the connected component, which shows that the same constant α_k is valid for all connected components. Thus (4.16) holds for all $\lambda \in \mathbb{C}$.

The exact value of α_k can then be determined by expanding (4.16) for large λ . Suppose for example that $k < 0$. Then by (3.1) and (3.3)

$$|w_k(\lambda)| = \prod_{j=1}^{q+k} |z_j(\lambda)| = |a_{-q}|^{(q+k)/q} |\lambda|^{-(q+k)/q} \left(1 + \mathcal{O}(\lambda^{-1/q})\right)$$

as $\lambda \rightarrow \infty$. Thus

$$-\log |w_k(\lambda)| = \frac{q+k}{q} \log |\lambda| - \frac{q+k}{q} \log |a_{-q}| + \mathcal{O}(\lambda^{-1/q}). \quad (4.17)$$

Since

$$\int \log |\lambda - x| d\mu_k(x) = \log |\lambda| \mu_k(\Gamma_k) + o(1) = \frac{q+k}{q} \log |\lambda| + o(1), \quad (4.18)$$

as $\lambda \rightarrow \infty$, the value (4.11) for α_k follows from (4.16), (4.17), and (4.18). The argument for $k > 0$ is similar. This completes the proof of the proposition. \square

To prove part (c) of Theorem 2.3 we also need the following lemma.

Lemma 4.3. *Let $\vec{\nu}_1 = (\nu_{1,-q+1}, \dots, \nu_{1,p-1})$ and $\vec{\nu}_2 = (\nu_{2,-q+1}, \dots, \nu_{2,p-1})$ be two admissible vectors of measures. Then $J(\vec{\nu}_1 - \vec{\nu}_2)$ is well defined and*

$$J(\vec{\nu}_1 - \vec{\nu}_2) \geq 0, \quad (4.19)$$

with equality if and only if $\vec{\nu}_1 = \vec{\nu}_2$.

Proof. Since both $\vec{\nu}_1$ and $\vec{\nu}_2$ have finite energy, we find that $J(\vec{\nu}_1 - \vec{\nu}_2)$ is well defined. According to the alternative representation (2.12), we have

$$\begin{aligned} J(\vec{\nu}_1 - \vec{\nu}_2) &= \left(\frac{1}{q} + \frac{1}{p} \right) I(\nu_{1,0} - \nu_{2,0}) \\ &\quad + \sum_{k=1}^{q-1} k(k+1) I \left(\frac{\nu_{1,-q+k}}{k} - \frac{\nu_{2,-q+k}}{k} - \frac{\nu_{1,-q+k+1}}{k+1} + \frac{\nu_{2,-q+k+1}}{k+1} \right) \\ &\quad + \sum_{k=1}^{p-1} k(k+1) I \left(\frac{\nu_{1,p-k}}{k} - \frac{\nu_{2,p-k}}{k} - \frac{\nu_{1,p-k-1}}{k+1} + \frac{\nu_{2,p-k-1}}{k+1} \right). \end{aligned} \quad (4.20)$$

Using (2.6) and (2.8), we see that all terms in (4.20) are non-negative and therefore (4.19) holds.

Suppose now that $J(\vec{\nu}_1 - \vec{\nu}_2) = 0$. Then all terms in the right-hand side of (4.20) are zero, so that

$$\nu_{1,0} = \nu_{2,0}, \quad (4.21)$$

$$\frac{\nu_{1,-q+k}}{k} + \frac{\nu_{2,-q+k+1}}{k+1} = \frac{\nu_{1,-q+k+1}}{k+1} + \frac{\nu_{2,-q+k}}{k}, \quad \text{for } k = 1, \dots, q-1, \quad (4.22)$$

$$\frac{\nu_{1,p-k}}{k} + \frac{\nu_{2,p-k-1}}{k+1} = \frac{\nu_{1,p-k-1}}{k+1} + \frac{\nu_{2,p-k}}{k}, \quad \text{for } k = 1, \dots, p-1. \quad (4.23)$$

Using (4.21) in (4.22) with $k = q-1$, we find $\nu_{1,-1} = \nu_{2,-1}$. Proceeding inductively we then obtain from (4.22) that $\nu_{1,k} = \nu_{2,k}$ for all $k = -q+1, \dots, 0$. Similarly, from (4.21) and (4.23) it follows that $\nu_{1,k} = \nu_{2,k}$ for $k = 0, \dots, p-1$, so that $\vec{\nu}_1 = \vec{\nu}_2$ as claimed. \square

Now we are ready for the proof of Theorem 2.3.

Proof of Theorem 2.3. (a) In view of Proposition 4.1 it only remains to show that $\mu_k \in \mathcal{M}_e$ for every $k = -q+1, \dots, p-1$. The decay estimate (4.5) implies that

$$\int \log(1 + |\lambda|) d\mu_k(\lambda) < \infty.$$

The fact that $I(\mu_k) < +\infty$ follows from (4.10). Indeed,

$$I(\mu_k) = - \iint \log |\lambda - x| d\mu_k(x) d\mu_k(\lambda) = \int (\log |w_k(\lambda)| - \alpha_k) d\mu_k(\lambda)$$

and this is finite since μ_k is a finite measure on Γ_k with a density that decays as in (4.5) and $\log |w_k(\lambda)|$ is continuous on Γ_k and grows only as a constant times $\log |\lambda|$ as $\lambda \rightarrow \infty$. Thus $\vec{\mu}$ is admissible and part (a) is proved.

(b) According to (4.10) we have

$$\begin{aligned}
& 2 \int \log |\lambda - x| \, d\mu_k(x) - \int \log |\lambda - x| \, d\mu_{k+1}(\lambda) - \int \log |\lambda - x| \, d\mu_{k-1}(\lambda) \\
&= -2 \log |w_k(\lambda)| + 2\alpha_k + \log |w_{k+1}(\lambda)| - \alpha_{k+1} + \log |w_{k-1}(\lambda)| - \alpha_{k-1} \\
&= \log \left| \frac{w_{k+1}(\lambda)w_{k-1}(\lambda)}{w_k(\lambda)^2} \right| + 2\alpha_k - \alpha_{k+1} - \alpha_{k-1} \\
&= \log \left| \frac{z_{q+k+1}(\lambda)}{z_{q+k}(\lambda)} \right| + 2\alpha_k - \alpha_{k+1} - \alpha_{k-1}. \tag{4.24}
\end{aligned}$$

Since $|z_{q+k}(\lambda)| = |z_{q+k+1}(\lambda)|$ for $\lambda \in \Gamma_k$, we see from (4.24) that (2.10) holds with constant

$$l_k = 2\alpha_k - \alpha_{k-1} + \alpha_{k+1}. \tag{4.25}$$

Note that for $k = -q + 1$ and $k = p - 1$, we are using the convention that $\mu_{-q} = \mu_p = 0$, and we also have put $\alpha_{-q} = \alpha_p = 0$. This proves part (b).

(c) Let $\vec{\nu} = (\nu_{-q+1}, \dots, \nu_{p-1})$ be any admissible vector of measures. From the representation (2.13) we get

$$\begin{aligned}
J(\vec{\nu}) &= J(\vec{\mu} + \vec{\nu} - \vec{\mu}) \\
&= J(\vec{\mu}) + J(\vec{\nu} - \vec{\mu}) + 2 \sum_{j,k=-q+1}^{p-1} A_{jk} I(\mu_j, \nu_k - \mu_k). \tag{4.26}
\end{aligned}$$

Using (2.14), we find from (4.26)

$$J(\vec{\nu}) = J(\vec{\mu}) + J(\vec{\nu} - \vec{\mu}) + \sum_{k=-q+1}^{p-1} I(2\mu_k - \mu_{k-1} - \mu_{k+1}, \nu_k - \mu_k) \tag{4.27}$$

For each $k = -q + 1, \dots, p - 1$, we have

$$\begin{aligned}
& I(2\mu_k - \mu_{k-1} - \mu_{k+1}, \nu_k - \mu_k) \\
&= \int \left(\int \log |\lambda - x| \, d(2\mu_k - \mu_{k-1} - \mu_{k+1})(x) \right) d(\nu_k - \mu_k)(\lambda) \tag{4.28}
\end{aligned}$$

By (2.10) the inner integral in the right-hand side of (4.28) is constant for $\lambda \in \Gamma_k$. Since ν_k and μ_k are finite measures on Γ_k with $\nu_k(\Gamma_k) = \mu_k(\Gamma_k)$, we find from (4.28) that

$$I(2\mu_k - \mu_{k-1} - \mu_{k+1}, \nu_k - \mu_k) = 0, \quad \text{for } k = -q + 1, \dots, p - 1.$$

Then (4.27) shows that $J(\vec{\nu}) = J(\vec{\mu}) + J(\vec{\nu} - \vec{\mu})$, which by Lemma 4.3 implies that $J(\vec{\nu}) \geq J(\vec{\mu})$ and equality holds if and only if $\vec{\nu} = \vec{\mu}$. This completes the proof of Theorem 2.3. \square

5. PROOFS OF PROPOSITION 2.5 AND THEOREM 2.6

5.1. Proof of Proposition 2.5. We will now prove Proposition 2.5, which follows by a combinatorial argument.

Proof of Proposition 2.5. We prove (2.18) and (2.19) for $k > 0$. The case $k < 0$ is similar. Let us first expand the determinant in the definition of $P_{k,n}$

$$P_{k,n}(\lambda) = \det T_n(z^{-k}(a - \lambda)) = \sum_{\pi \in S_n} \prod_{j=1}^n (a - \lambda)_{j-\pi(j)+k}. \quad (5.1)$$

Here S_n denotes the set of all permutation on $\{1, \dots, n\}$. By the band structure of $T_n(z^{-k}(a - \lambda))$ it follows that we only have non-zero contributions from permutations π that satisfy

$$k - p \leq \pi(j) - j \leq q + k, \quad \text{for all } j = 1, \dots, n. \quad (5.2)$$

Define for $\pi \in S_n$,

$$N_\pi = \{j \mid \pi(j) = j + k\}. \quad (5.3)$$

and denote the number of elements of N_π by $|N_\pi|$. For each $\pi \in S_n$ we have that $\prod_{j=1}^n (a - \lambda)_{j-\pi(j)+k}$ is a polynomial in λ of degree at most $|N_\pi|$. So by (5.1)

$$d_{k,n} = \deg P_{k,n} \leq \max_{\pi} |N_\pi| \quad (5.4)$$

where we maximize over permutations $\pi \in S_n$ satisfying (5.2).

Let $\pi \in S_n$ satisfying (5.2). We prove (2.18) by giving an upper bound for $|N_\pi|$. Since $\sum_{j=1}^n (\pi(j) - j) = 0$ we obtain

$$\sum_{j=1}^n (\pi(j) - j)_+ = \sum_{j=1}^n (j - \pi(j))_+, \quad (5.5)$$

where $(\cdot)_+$ is defined as $(a)_+ = \max(0, a)$ for $a \in \mathbb{R}$. Each $j \in N_\pi$ gives a contribution k to the left-hand side of (5.5). Therefore the left-hand side is at least $k|N_\pi|$. By (5.2) we have that each term in the right hand side is at most $p - k$. Moreover, there are at most $n - |N_\pi|$ non-zero terms in this sum. Combining this with (5.5) leads to

$$k|N_\pi| \leq \sum_{j=1}^n (\pi(j) - j)_+ = \sum_{j=1}^n (j - \pi(j))_+ \leq (n - |N_\pi|)(p - k). \quad (5.6)$$

Hence, if π is a permutation satisfying (5.2)

$$|N_\pi| \leq \frac{n(p - k)}{p}. \quad (5.7)$$

Now (2.18) follows by combining (5.7) and (5.4).

To prove (2.19), we assume that $n \equiv 0 \pmod{p}$. We claim that there exists a unique π such that equality holds in (5.7). Then equality holds in both

inequalities of (5.6) and the above arguments show that this can only happen if

$$\pi(j) = j + k, \quad \text{or} \quad \pi(j) = j - p + k, \quad (5.8)$$

for every $j = 1, \dots, n$. We claim that there exists a unique such permutation, namely

$$\pi(j) = \begin{cases} j + k, & \text{if } j \equiv 1, \dots, (p - k) \pmod{p}, \\ j - p + k, & \text{if } j \equiv (p - k + 1), \dots, p \pmod{p}. \end{cases} \quad (5.9)$$

To see this let π be a permutation satisfying (5.8). The numbers $1, \dots, p - k$ can not satisfy $\pi(j) = j - p + k$ and thus satisfy $\pi(j) = j + k$. On the other hand, the numbers $1, \dots, k$ can not be the image of numbers j satisfying $\pi(j) = j + k$, and thus $\pi(j) = j - p + k$ for $j = p - k + 1, \dots, p$. So (5.9) holds for $j = 1, \dots, p$. This means in particular that the restriction of π to $\{p + 1, \dots, n\}$ is again a permutation, but now on $\{p + 1, \dots, n\}$. By the same arguments we then find that (5.9) holds for $j = p + 1, \dots, 2p$, and so on. The result is that (5.9) is indeed the only permutation that satisfies (5.8).

Finally, a straightforward calculation shows that the coefficient of $\lambda^{(p-k)n/p}$ in $\prod_{j=1}^n (a - \lambda)_{j-\pi(j)+k}$ with π as in (5.9) is nonzero and given by (2.19). This proves the proposition. \square

5.2. Proof of Theorem 2.6. Before we start with the proof of Theorem 2.6 we first prove the following proposition concerning the asymptotics for $P_{k,n}$ for $n \rightarrow \infty$.

Proposition 5.1. *Let $M_k = \{q + k + 1, \dots, p + q\}$. We have that*

$$P_{k,n}(\lambda) = (w_{M_k}(\lambda))^n C_{M_k}(\lambda) (1 + \mathcal{O}(\exp(-c_K n))), \quad n \rightarrow \infty, \quad (5.10)$$

uniformly on compact subsets K of $\mathbb{C} \setminus \Gamma_k$. Here c_K is a positive constant depending on K .

Proof. First rewrite (2.22) as

$$P_{k,n}(\lambda) = (w_{M_k}(\lambda))^n C_{M_k}(\lambda) (1 + R_{k,n}(\lambda)). \quad (5.11)$$

with $R_{k,n}$ defined by

$$R_{k,n}(\lambda) = \sum_{M \neq M_k} \frac{(w_M(\lambda))^n C_M(\lambda)}{(w_{M_k}(\lambda))^n C_{M_k}(\lambda)}. \quad (5.12)$$

Let K be a compact subset of $\mathbb{C} \setminus \Gamma_k$. If K does not contain branch points then there exists $A, B > 0$ such that

$$A < |C_M(\lambda)| < B \quad (5.13)$$

for all $\lambda \in K$ and M . Moreover, we have

$$\left| \frac{w_M(\lambda)}{w_{M_k}(\lambda)} \right| \leq \left| \frac{z_{q+k}(\lambda)}{z_{q+k+1}(\lambda)} \right| \leq \sup_{\lambda \in K} \left| \frac{z_{q+k}(\lambda)}{z_{q+k+1}(\lambda)} \right| < 1, \quad (5.14)$$

for all $\lambda \in K$ and $M \neq M_k$. Therefore one readily verifies from (5.11) that there exist c_K such that $|R_{k,n}(\lambda)| \leq \exp(-c_K n)$ for all $\lambda \in K$ and n large enough. This proves the statement in case K does not contain branch points.

Suppose that K does contain branch points. Without loss of generality we can assume that all branch points lie in the interior of K (otherwise we replace K by a bigger compact set). The boundary ∂K of K is a compact set with no branch points and therefore (5.10) holds for ∂K by the above arguments. Since w_{M_k} and C_{M_k} are analytic in K , we find by (5.11) that $R_{k,n}$ is analytic in K . The maximum modulus principle for analytic functions states that $\sup_{z \in K} |R_{k,n}(z)| = \sup_{z \in \partial K} |R_{k,n}(z)|$ and thereby we obtain that (5.10) also holds for K with the same constant $c_K = c_{\partial K}$. \square

We now state two particular consequences of (5.10).

Corollary 5.2. *Let $k \in \{-q+1, \dots, p-1\}$. For every compact set $K \subset \mathbb{C} \setminus \Gamma_k$ we have that $\mu_{k,n}(K) = 0$ for n large enough.*

Proof. Let K be a compact subset of $\mathbb{C} \setminus \Gamma_k$. By (5.10) it follows that $P_{k,n}$ has no zeros in K for large n . Since $n\mu_{k,n}(K)$ equals the number of zeros of $P_{k,n}$ in K the corollary follows. \square

Corollary 5.3. *Let $k \in \{-q+1, \dots, p-1\}$. We have that*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{C}} \frac{d\mu_{k,n}(x)}{x - \lambda} = \int_{\Gamma_k} \frac{d\mu_k(x)}{x - \lambda}, \quad (5.15)$$

uniformly on compact subsets of $\mathbb{C} \setminus \Gamma_k$.

Proof. Let K be a compact subset of $\mathbb{C} \setminus \Gamma_k$. Note that

$$\int \frac{d\mu_{k,n}(x)}{x - \lambda} = \frac{1}{n} \sum_{\lambda_i \in \text{sp}_k T_n(a)} \frac{1}{\lambda_i - \lambda} = -\frac{P'_{k,n}(\lambda)}{nP_{k,n}(\lambda)}, \quad (5.16)$$

for all $\lambda \in K$. With M_k and c_K as in Proposition 5.1 we obtain from (5.10) that

$$\frac{P'_{k,n}(\lambda)}{nP_{k,n}(\lambda)} = \frac{w'_{M_k}(\lambda)}{w_{M_k}(\lambda)} + \mathcal{O}(1/n), \quad n \rightarrow \infty, \quad (5.17)$$

uniformly on K . Let us rewrite the right-hand side of (5.17). By expanding both sides of $z^q(a(z) - \lambda) = a_p \prod_{j=1}^{p+q} (z - z_j(\lambda))$ and collecting the constant terms we obtain

$$\prod_{j=1}^{p+q} (-z_j(\lambda)) = \frac{a_{-q}}{a_p}. \quad (5.18)$$

Since $\lambda \notin \Gamma_k$, we can split this product in two parts, take the logarithmic derivative and use (3.3) and (2.23) to obtain

$$0 = \sum_{j=1}^{q+k} \frac{z'_j(\lambda)}{z_j(\lambda)} + \sum_{j=q+k+1}^{p+q} \frac{z'_j(\lambda)}{z_j(\lambda)} = \frac{w'_k(\lambda)}{w_k(\lambda)} + \frac{w'_{M_k}(\lambda)}{w_{M_k}(\lambda)}. \quad (5.19)$$

Combining (5.16), (5.17) and (5.19), we obtain

$$\lim_{n \rightarrow \infty} \int \frac{d\mu_{k,n}(x)}{x - \lambda} = \frac{w'_k(\lambda)}{w_k(\lambda)} \quad (5.20)$$

uniformly on K . Then (5.15) follows from (5.20) and (4.9). \square

Now we are ready for the proof of Theorem 2.6.

Proof of Theorem 2.6.

First we prove (2.21). By Proposition 2.5 and the fact that $\vec{\mu}$ is admissible, we get (see (2.8))

$$\mu_{k,n}(\mathbb{C}) = \frac{1}{n} \deg P_{k,n} \leq \mu_k(\mathbb{C}), \quad (5.21)$$

for every $n \in \mathbb{N}$.

Let $C_0(\mathbb{C})$ be the Banach space of continuous functions on \mathbb{C} that vanish at infinity. The dual space $C_0(\mathbb{C})^*$ of $C_0(\mathbb{C})$ is the space of regular complex Borel measures on \mathbb{C} . By (5.21) the sequence $(\mu_{k,n})_{n \in \mathbb{N}}$ belongs to the ball in $C_0(\mathbb{C})^*$ centered at the origin with radius $\mu_k(\mathbb{C})$, which is weak* compact by the Banach-Alaoglu theorem. Let $\mu_{k,\infty}$ be the limit of a weak* convergent subsequence of $(\mu_{k,n})_{n \in \mathbb{N}}$.

By weak* convergence and Corollary 5.2 we obtain that $\mu_{k,\infty}$ is supported on Γ_k . Combining this with (5.15) and the weak* convergence leads to

$$\frac{1}{2\pi i} \int_{\Gamma_k} \frac{d\mu_k(x)}{x - \lambda} = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{d\mu_{k,\infty}(x)}{x - \lambda}, \quad (5.22)$$

for every $\lambda \in \mathbb{C} \setminus \Gamma_k$. The integrals in (5.22) are known in the literature as the Cauchy transforms of the measures μ_k and $\mu_{k,\infty}$. The Cauchy transform on Γ_k is an injective map that maps measures on Γ_k to functions that are analytic in $\mathbb{C} \setminus \Gamma_k$ (one can find explicit inversion formulae, see for example the arguments in [9, Theorem II.1.4] or the Stieltjes-Perron inversion formula in the special case $\Gamma_k \subset \mathbb{R}$). Thus it follows from (5.22) that $\mu_{k,\infty} = \mu_k$. Therefore

$$\lim_{n \rightarrow \infty} \mu_{k,n} = \mu_k \quad (5.23)$$

in the sense of weak* convergence in $C_0(\mathbb{C})^*$. Thus (2.21) holds if ϕ is a continuous function that vanishes at infinity.

From (5.21) and (5.23) it also follows that

$$\lim_{n \rightarrow \infty} \mu_{k,n}(\mathbb{C}) = \mu_k(\mathbb{C}), \quad (5.24)$$

Then the sequence $(\mu_{k,n})_{n \in \mathbb{N}}$ is tight. That is, for every $\varepsilon > 0$ there exists a compact K such that $\mu_{k,n}(\mathbb{C} \setminus K) < \varepsilon$ for every $n \in \mathbb{N}$. By a standard approximation argument one can now show that (2.21) holds for every bounded continuous function ϕ on \mathbb{C} .

Having (2.21) and Proposition 5.1, we can prove (2.20) as in [1, Theorem 11.17]. Indeed, the sets $\liminf_{n \rightarrow \infty} \text{sp}_k T_n(a)$ and $\limsup_{n \rightarrow \infty} \text{sp}_k T_n(a)$ equal the support of μ_k , which is Γ_k . \square

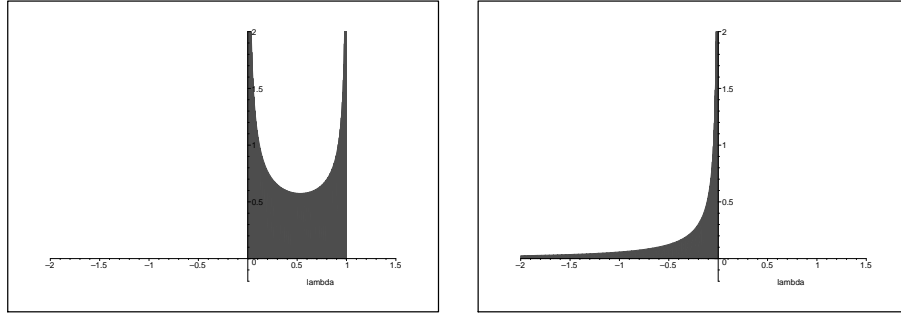


FIGURE 2. Illustration for Example 1: The densities of the measures μ_0 (left) and μ_1 (right) for $a = \frac{4(z+1)^3}{27z}$.

6. EXAMPLES

6.1. **Example 1.** As a first example consider the symbol a defined by

$$a(z) = \frac{4(z+1)^3}{27z}. \quad (6.1)$$

In this case we have $p = 2$ and $q = 1$. So we obtain two contours Γ_0 and Γ_1 with two associated measures μ_0 and μ_1 . This example appeared in [3], in which the authors gave explicit expressions for Γ_0 and μ_0 . The following proposition also contains expressions for Γ_1 and μ_1 . In what follows we take the principal branches for all fractional powers.

Proposition 6.1. *With a as in (6.1), we have that $\Gamma_0 = [0, 1]$ and*

$$d\mu_0(\lambda) = \frac{\sqrt{3}}{4\pi} \frac{(1 + \sqrt{1-\lambda})^{1/3} + (1 - \sqrt{1-\lambda})^{1/3}}{\lambda^{2/3}\sqrt{1-\lambda}} d\lambda. \quad (6.2)$$

Moreover, $\Gamma_1 = (-\infty, 0]$ and

$$d\mu_1(\lambda) = \frac{\sqrt{3}}{4\pi} \frac{(1 + \sqrt{1-\lambda})^{1/3} - (\sqrt{1-\lambda} - 1)^{1/3}}{(-\lambda)^{2/3}\sqrt{1-\lambda}} d\lambda. \quad (6.3)$$

Proof. A straightforward calculation shows that $\lambda = 0$ and $\lambda = 1$ are the branch points.

Let $\lambda \in \Gamma_0 \cup \Gamma_1$ and assume that λ is not a branch point. There exist $y_1, y_2 \in \mathbb{C}$ such that $y_1 \neq y_2$, $|y_1| = |y_2|$ and $a(y_1) = a(y_2) = \lambda$. Then it follows from (6.1) that $|y_1 + 1| = |y_2 + 1|$. Therefore y_1 and y_2 are intersection points of a circle centered at -1 and a circle centered at the origin. Since $y_1 \neq y_2$, this means that $y_1 = \overline{y_2}$ and therefore $\lambda = a(y_1) = a(\overline{y_2}) = \overline{a(y_2)} = \overline{\lambda}$, so that $\lambda \in \mathbb{R}$. A further investigation shows that $a(z) - \lambda$ has 3 different real zeros if $\lambda > 1$. If $\lambda < 1$ and $\lambda \neq 0$ then $a(z) - \lambda$ has precisely 1 real zero and 2 conjugate complex zeros. Therefore, $\Gamma_0 \cup \Gamma_1 = (-\infty, 1]$.

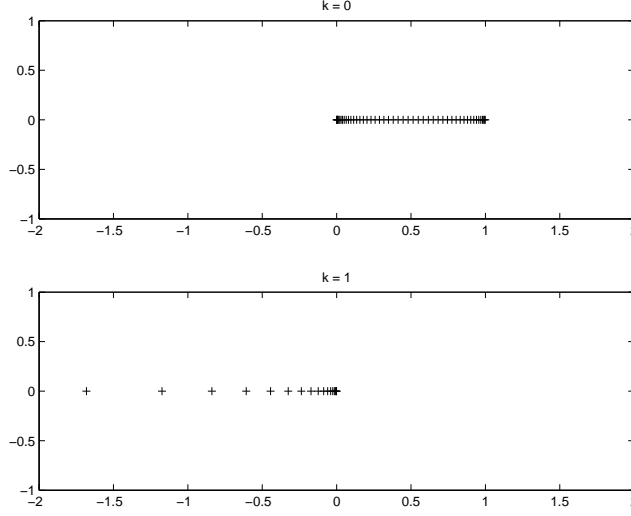


FIGURE 3. Illustration for Example 1: The spectrum $\text{sp } T_{50}(a)$ (top) and the generalized spectrum $\text{sp}_1 T_{50}(a)$ (bottom), for the symbol $a = \frac{4(z+1)^3}{27z}$.

Now we will show that $\Gamma_0 = [0, 1]$ and $\Gamma_1 = (-\infty, 0]$. By Cardano's formula the solutions of the algebraic equation $a(z) = \lambda$ are given by

$$z_j(\lambda) = -1 - \frac{3\lambda^{1/3}}{2} \left(\omega^j \left(1 + (1 - \lambda)^{1/2} \right)^{1/3} + \omega^{-j} \left(1 - (1 - \lambda)^{1/2} \right)^{1/3} \right), \quad (6.4)$$

for $\lambda \in [0, 1]$ and

$$z_j(\lambda) = -1 + \frac{3(-\lambda)^{1/3}}{2} \left(\omega^{j+2} \left(1 + (1 - \lambda)^{1/2} \right)^{1/3} - \omega^{-j-2} \left((1 - \lambda)^{1/2} - 1 \right)^{1/3} \right), \quad (6.5)$$

for $\lambda \in (-\infty, 0]$. Here $\omega = e^{2\pi i/3}$. One can check that $|z_1(\lambda)| = |z_2(\lambda)| < |z_3(\lambda)|$ for $\lambda \in (0, 1]$ and $|z_1(\lambda)| < |z_2(\lambda)| = |z_3(\lambda)|$ for $\lambda \in (-\infty, 0]$. Moreover, for $\lambda = 0$ we have $z_1(0) = z_2(0) = z_3(0) = -1$. Therefore $\Gamma_0 = [0, 1]$ and $\Gamma_1 = (-\infty, 0]$.

The density (6.2) was already given in [3] and (6.3) follows in a similar way. \square

In Figure 2 we plotted the densities of μ_0 and μ_1 . Note that, due to the interaction between μ_0 and μ_1 in the energy functional, there is more mass of μ_0 near 0 than near 1. We also see that the singularities of the densities for μ_0 and μ_1 are of order $\mathcal{O}(|\lambda|^{-2/3})$ for $\lambda \rightarrow 0$, whereas the typical nature of a singularity in each of the measures is a square root singularity. The stronger singularity is due to the fact that $a(z) - \lambda$ has a triple root for $\lambda = 0$.

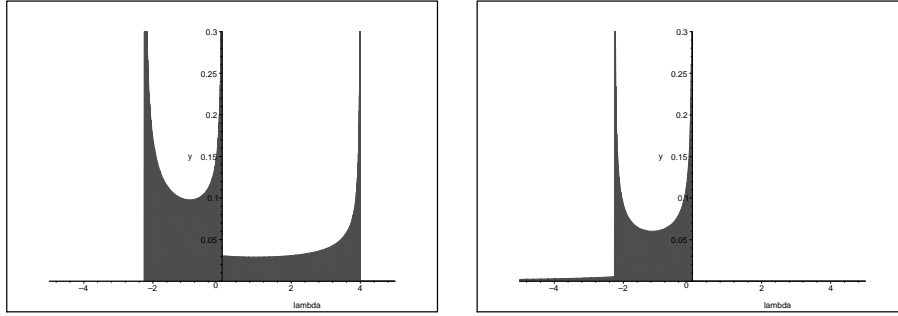


FIGURE 4. Illustration for Example 2: The densities of the measures μ_0 (left) and $\mu_1 = \mu_{-1}$ (right) for $a(z) = z^2 + z + z^{-1} + z^{-2}$.

In Figure 3 we plotted the eigenvalues and generalized eigenvalues for $n = 50$. It is known that the eigenvalues are simple and positive [3, §2.3], which we also see in Figure 3.

6.2. Example 2. For the symbol a defined by

$$a(z) = z^2 + z + z^{-1} + z^{-2}. \quad (6.6)$$

we have $p = q = 2$. From the symmetry $a(1/z) = a(z)$ it follows that $\Gamma_{-1} = \Gamma_1$ and $\mu_{-1} = \mu_1$.

The interesting feature of this example is that the contours Γ_0 and $\Gamma_{\pm 1}$ overlap. To be precise, the interval $(-9/4, 0)$ is contained in all three contours Γ_{-1}, Γ_0 and Γ_1 . This can be most easily seen by investigating the image of the unit circle under a . Consider

$$a(e^{it}) = 2 \cos 2t + 2 \cos t, \quad \text{for } t \in [0, 2\pi). \quad (6.7)$$

A straightforward analysis shows that for every $\lambda \in (-9/4, 0)$, the equation $a(e^{it}) = \lambda$ has four different solutions for t in $[0, 2\pi)$. This means that the four solutions of the equation $a(z) = \lambda$ are on the unit circle, and so in particular have the same absolute value.

The equation $a(z) - \lambda = 0$ can be explicitly solved by introducing the variable $y = z + 1/z$. In exactly the same way as in the previous example one can obtain the limiting measures. We will not give the explicit formulas, but only plot the densities in Figure 4. The branch points are $\lambda = -9/4$, $\lambda = 0$ and $\lambda = 4$. The contours are given by

$$\Gamma_0 = [-9/4, 4], \quad \Gamma_{-1} = \Gamma_1 = (-\infty, 0]. \quad (6.8)$$

The densities have singularities at the branch points in the interior of their supports. The singularities are only felt at one side of the branch points. Consider first μ_0 , whose density has a singularity at 0. However the limiting value when 0 is approached from the positive real axis is finite. The change in behavior of μ_0 has to do with the fact that z_1 is analytic on $(0, 4)$ but not

on $(-9/4, 0)$. Therefore we find by (1.12) that

$$d\mu_0(\lambda) = \frac{1}{2\pi i} \left(\frac{z'_{1+}(\lambda)}{z_{1+}(\lambda)} + \frac{z'_{2+}(\lambda)}{z_{2+}(\lambda)} - \frac{z'_{1-}(\lambda)}{z_{1-}(\lambda)} - \frac{z'_{2-}(\lambda)}{z_{2-}(\lambda)} \right) d\lambda \quad (6.9)$$

on $(-9/4, 0)$, and

$$d\mu_0(\lambda) = \frac{1}{2\pi i} \left(\frac{z'_{2+}(\lambda)}{z_{2+}(\lambda)} - \frac{z'_{2-}(\lambda)}{z_{2-}(\lambda)} \right) d\lambda \quad (6.10)$$

on $(0, 4)$.

For $\mu_{-1} = \mu_1$ a similar phenomenon happens at $\lambda = -9/4$. This is a consequence of the fact that z_1 has an analytic continuation into z_2 when we cross $(-\infty, -9/4)$, but it has an analytic continuation into z_4 when we cross $(-9/4, 0)$.

6.3. Example 3. As a final example, consider the symbol

$$a(z) = z^p + z^{-q}, \quad (6.11)$$

with $p, q \geq 1$ and $\gcd(p, q) = 1$. This example appeared in [10], where the authors mentioned that Γ_0 is given by the star

$$\Gamma_0 = \{r\omega^j \mid j = 1, \dots, p+q, 0 \leq r \leq R\} \quad (6.12)$$

with $\omega = e^{2\pi i/(p+q)}$ and $R = (p+q)p^{-p/(p+q)}q^{-q/(p+q)}$. The other contours also have a star shape, namely

$$\Gamma_k = \{(-1)^k r\omega^j \mid j = 1, \dots, p+q, 0 \leq r < \infty\} \quad (6.13)$$

for $k \neq 0$. Note that the star Γ_k for $k \neq 0$ is unbounded.

In Figure 5 we plotted the eigenvalues and the generalized eigenvalues for $p = 2$, $q = 3$ and $n = 50$. All the (generalized) eigenvalues appear to lie exactly on the contours. In the special case $p = 1$ it is known that the eigenvalues of $T_n(a)$ lie indeed precisely on the star (6.12) and are all simple (possibly except for 0) [4, Theorem 3.2], see also [6] for a connection to Chebyshev-type quadrature.

6.4. Numerical stability. In Figure 3 and Figure 5 the eigenvalues and the generalized eigenvalues of $T_{50}(a)$ were computed numerically. To control the stability of the numerical computation of the eigenvalues one needs to analyze the pseudo-spectrum. For banded Toeplitz matrices the pseudo-spectrum is well understood [12, Th. 7.2]. To this date, a similar analysis of the pseudo-spectrum for the matrix pencil $(T_n(z^{-k}a), T_n(z^{-k}))$ has not been carried out. See [12, §X.45] for some remarks on the pseudo-spectrum for the generalized eigenvalue problem.

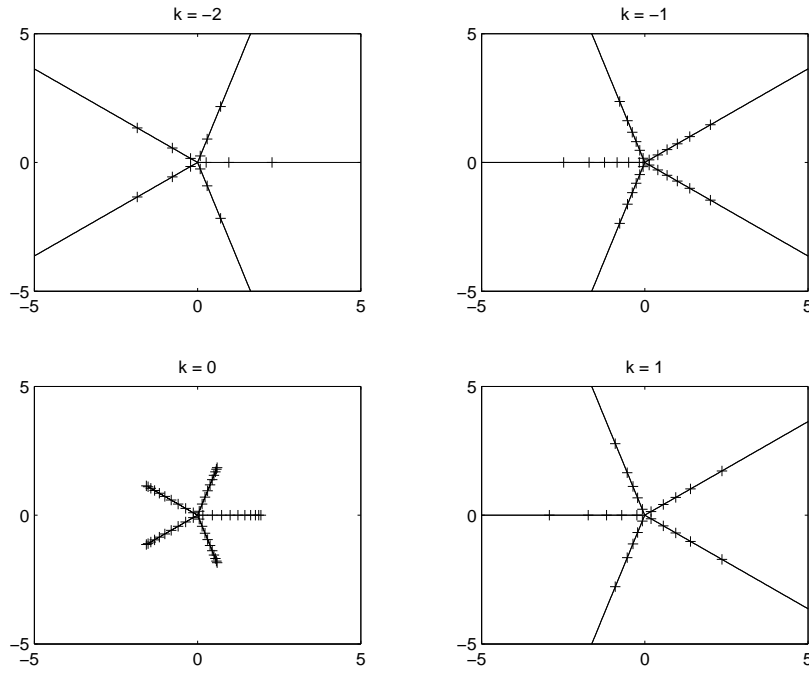


FIGURE 5. Illustration for Example 3: The contours Γ_k and the eigenvalues and generalized eigenvalues for $T_{50}(a)$ for the symbol $a = z^2 + z^{-3}$.

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